BIFURCATION ANALYSIS OF FLIGHT CONTROL SYSTEMS

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Abstract: High performance systems, like the F-16, when pushed to their limits encounter qualitative changes in control system properties like loss of controllability or observability. This work identifies and characterizes bifurcations occurring in a nonlinear six degree of freedom F-16 in two scenarios - straight and level flight and in a coordinated turn. Phenomena such as stall, tumbling and spin-roll departure were observed around bifurcation points. This work provides a basis for a formal understanding of how aircraft depart from controlled flight, it is a prerequisite for the systematic design of recovery strategies, and it will contribute to the design of reconfigurable control of impaired aircraft.

Keywords: flight control, departure, bifurcation, F-16

1. INTRODUCTION

An aircraft can encounter sudden changes in behavior when it executes high performance maneuvers. In order to design control systems that encompass a large flight envelope and ensure demanding performance criteria, we must be aware of the limitations of the system and understand the characteristics of the controlled aircraft when it approaches these limiting conditions. The limiting points include bifurcation points of the underlying mathematical model. Although bifurcations have been studied extensively from a dynamical perspective, those arising from a control system perspective have not received much attention. Bifurcations in control systems are associated with regulating certain variables of the system. At bifurcation points of the equilibrium equations, the linearized system always has degeneracies in the zero dynamics.

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There have been several studies intending to understand and predict nonlinear phenomena like stall, wing rock, spin in aircraft. An investigation of the high angle of attack behavior of the same aircraft analyzed in this work, was conducted on the Langley differential maneuvering simulator (Nguyen *et al.*, 1979). Various maneuvers were initiated to investigate issues like controllability and departure susceptibility. A pioneering work in the study of nonlinear phenomena by a continuation approach based on the aircraft's mathematical model is (Carrol and Mehra, 1982). The authors were able to identify trim conditions corresponding to the onset of spin and wing rock. Further examples along this line of research can be found in (Jahnke and Culick, 1994) and (Liaw *et al.*, 2003). The bifurcation analysis of the controller augmented aircraft was considered in (Avanzini and Matteis, 1997) and (Gibson *et al.*, 1998). The ultimate goal of all the above research is to design control systems that circumvent or alleviate the undesirable nonlinear phenomena. The bifurcation analysis of the mathematical models is carried out from a purely dynamical system perspective, and thus the bifurcation parameters correspond to control surfaces positions. This analysis, may not correspond to what the aircraft encounters in practice and does not reveal control degeneracies that accompany the bifurcation.

In this work we identify and characterize the bifurcation points for a F-16 fighter aircraft in two conditions; straight and level flight and coordinated turn at constant altitude. Three bifurcation points are identified in each case. The bifurcations are associated with regulation of the aircraft's speed, flight path and orientation. The speed of the aircraft and the values of the longitudinal variables at which the bifurcations occur do not differ significantly between the two cases. At the bifurcation points in level flight, there is a transmission zero at the origin, the aircraft is uncontrollable and there are dependent inputs. In the bifurcations associated with the coordinated turn, the aircraft loses observability and has dependent outputs, in addition to reasons for the bifurcations in level flight. Most bifurcation points are associated with aircraft stall. Tumbling stall and an emergent spin that transitions to a roll departure is also observed.

A nonlinear high fidelity F-16 mathematical model and an accompanying simulation model were developed using *Mathematica*. The model is valid in a large flight envelope because the the aerodynamic forces and moments are based on a polynomial modelling technique developed by Morelli (Morelli, 1998). This allows for wide variations in the angle of attack, side slip angle and control surface deflections. The model is described more completely in (Thomas *et al.*, 2004).

The bifurcation points and the dependence of the system variables on the parameters are illustrated graphically by bifurcation diagrams. There are well known software packages (Doedel and Wang, 1995) that analyze bifurcations in dynamical systems. These packages, however, are not directly amenable to analyze bifurcations in the context of control systems. The tools we use are basically the Newton-Raphson and the Newton-Raphson-Seydel (Seydel, 1994) methods. There are several subtleties when bifurcation points are clustered together, when many eigenvalues are near the origin or when the eigenvalues nearest the origin are complex. These issues have been addressed in the paper.

The rest of the paper is organized as follows. Bifurcation analysis in the context of control systems, and the tools used to carry out the analysis are discussed in Section 2. The nonlinear six degree of freedom F-16 model is presented in Section 3. Section 4 details the controller design. Sections 5 and 6 summarize the results of the bifurcation analysis for the level flight condition and the coordinated turn condition respectively. Section 7 is the conclusion.

2. BIFURCATION ANALYSIS OF CONTROL SYSTEMS

Consider a parameter dependent, nonlinear control system given by

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}, \mu) \\ \mathbf{z} &= \mathbf{r}(\mathbf{x}, \mu) \end{aligned} \quad (1)$$

where $\mathbf{x} \in R^n$ are the states, $\mathbf{u} \in R^p$ are the control inputs, $\mathbf{z} \in R^r$ are the regulated variables and $\mu \in R$ is any parameter. The parameter could be a physical variables like the weight of the aircraft or the center of gravity location; or a regulated variable like velocity, flight path angle, altitude or roll angle; or even a stuck control surface. The regulator problem is solvable only if $p \geq r$. Since the number of controls can always be reduced we henceforth assume $p = r$.

A triple $(\mathbf{x}^\star, \mathbf{u}^\star, \mu^\star)$ is an *equilibrium* point of (1) if

$$\boldsymbol{\chi}(\mathbf{x}^\star, \mathbf{u}^\star, \mu^\star) := \begin{pmatrix} \mathbf{f}(\mathbf{x}^\star, \mathbf{u}^\star, \mu^\star) \\ \mathbf{r}(\mathbf{x}^\star, \mu^\star) \end{pmatrix} = \mathbf{0} \quad (2)$$

Definition 2.1. (Kwatny *et al.*, 1991) An equilibrium point $(\mathbf{x}^\star, \mathbf{u}^\star, \mu^\star)$ is *regular* if there is a neighborhood of $\mu^\star$ on which there exist unique, continuously differentiable functions $\overline{\mathbf{x}}(\mu)$, $\overline{\mathbf{u}}(\mu)$ satisfying

$$\boldsymbol{\chi}(\overline{\mathbf{x}}(\mu), \overline{\mathbf{u}}(\mu), \mu) = \mathbf{0} \quad (3)$$

If an equilibrium point is not a regular point it is a *static bifurcation point.* The Implicit Function Theorem implies that an equilibrium point is a bifurcation point only if $\det J = 0$. The Jacobian J is given by

$$J = [D_{\mathbf{x}}\boldsymbol{\chi}(\mathbf{x}^\star, \mathbf{u}^\star, \mu^\star) \quad D_{\mathbf{u}}\boldsymbol{\chi}(\mathbf{x}^\star, \mathbf{u}^\star, \mu^\star)] \quad (4)$$

Now, let A, B, C, D denote the linearization at $(\mathbf{x}^\star, \mathbf{u}^\star, \mu^\star)$ of (1) with output $\mathbf{z}$ so that

$$J = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$$

Then we have the following theorem for a static bifurcation point.

Theorem 2.2. (Kwatny *et al.*, 1991) An equilibrium point $(\mathbf{x}^\star, \mathbf{u}^\star, \mu^\star)$ is a static bifurcation point only if

$$Im \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \neq R^{n+r} \quad (5)$$

Recall that the system matrix is

$$P(\lambda) = \begin{pmatrix} \lambda I - A & B \\ -C & 0 \end{pmatrix}$$

From this observation, *necessary* conditions for a static bifurcation point can be obtained as follows (Kwatny *et al.*, 2003):

Theorem 2.3. The equilibrium point $(\mathbf{x}^\star, \mathbf{u}^\star, \mu^\star)$ is a static bifurcation point of (1) only if one of the following conditions is true for its linearization:

1. there is a transmission zero at the origin,
2. there is an uncontrollable mode with zero eigenvalue,
3. there is an unobservable mode with zero eigenvalue,
4. it has insufficient independent controls,
5. it has redundant regulated variables.

There is a fundamental difference between bifurcation analysis of dynamical systems and control systems. As seen above, the behavioral aspects at the bifurcation points of control systems involve issues of system controllability, observability, et cetera, which are nonexistent for dynamical system bifurcation analysis.

2.1 Bifurcation diagrams

To obtain the bifurcation diagrams, which is essentially a locus of the equilibrium points, we start at a known equilibrium condition and employ the Newton-Raphson (NR) method in a continuation process. Of course, it fails to converge as a bifurcation point is approached. To resolve this issue we replace the NR method by the Newton-Raphson-Seydel (NRS) method (Seydel, 1994) at the point where the former breaks down. The NRS method is essentially the NR method applied to a modified set of equations, namely

$$\boldsymbol{\chi}(\mathbf{x}, \mathbf{u}, \mu) = \mathbf{0} \tag{6}$$

$$J\,\widetilde{\mathbf{v}} = \lambda \widetilde{\mathbf{v}} \tag{7}$$

$$\|\widetilde{\mathbf{v}}\| = 1 \tag{8}$$

The idea is to evaluate the Jacobian at the point where the NR method fails and compute its eigenvalues and eigenvectors. The initial approximation of λ and $\widetilde{\mathbf{v}}$ in (7) and (8) is chosen as the smallest eigenvalue of J and its corresponding eigenvector, respectively. In equations (6) - (8), λ is treated as a parameter and is made to approach zero. $\lambda = 0$ corresponds to a bifurcation point. Equations (7) and (8), with $\lambda = 0$, require that the Jacobian J be singular at the bifurcation point - a necessary condition for bifurcation. After reaching the bifurcation point, we can progress along the bifurcation curve by moving λ away from the origin in the opposite direction from which we arrived at the bifurcation point. Once we are sufficiently far away from any bifurcation point, we can revert to the NR method.

The above approach assumes that the smallest eigenvalue of the Jacobian, evaluated at the point where NR breaks down, is the one that eventually goes to zero at the bifurcation point. A dilemma occurs if there are two bifurcation points close to the origin. Also if the eigenvalues closest to the origin is a complex pair we do not know a priori the path they will take to the real axis so that one of them can eventually wind up at the origin. In such instances it is advisable to choose an eigenvalue, whose locus is well established even if it is not approaching the origin, and its corresponding eigenvector. As we near the bifurcation point, the smallest eigenvalue unambiguously approaches zero and we can revert to it.

It should also be noted that an eigenvalue of the Jacobian may become small just by virtue of the proximity of the present point to a bifurcation point on another portion of the bifurcation curve. In such instances, allowing λ to converge zero could result in a solution on the other branch of the curve. Also, in cases where there are two consecutive bifurcation points on the bifurcation curve, as we leave one and approach the other the smallest eigenvalue may move away from the origin and approach the origin from the same side. In both these cases, as discussed above, it is better to work with an eigenvalue with a well established locus.

3. SIX DEGREE OF FREEDOM F-16 MODEL

As noted above, the aircraft is treated as a 6-dof rigid body. The body fixed reference frame is located at the reference center of gravity location with the X, Y and Z axes in the forward, right wing and downward direction respectively. The position (x, y, z) and orientation (ϕ, θ, ψ) of this reference frame with respect to a earth fixed frame, together with the angular velocities (p, q, r) and the linear velocities (u, v, w) with respect to X, Y and Z respectively, completely characterize the motion of the airframe.

We consider a model with five control inputs, namely thrust T, left δ_{el} and right δ_{er} elevator, aileron δ_a and rudder δ_r. The control surface angles are limited as follows: elevators $|\delta_{er}|, |\delta_{el}| \le 0.436$ rad (25°), aileron $|\delta_a| \le 0.375$ rad (21.5°), and rudder $|\delta_r| \le 0.524$ rad (30°).

The nondimensional aerodynamic force and moment coefficients are expressed as multivariate nonlinear functions and were adapted from (Morelli, 1998). Additional physical data was obtained from (Garza and Morelli, 2003) and (Nguyen *et al.*, 1979).

Then, we have the state space system

$$\dot{\mathbf{X}} = f(\mathbf{X}, \mathbf{u}) \tag{9}$$

where $\mathbf{X} = [\phi\ \theta\ \psi\ x\ y\ z\ p\ q\ r\ V\ \alpha\ \beta]^T$. The input vector is $\mathbf{u} = [T, \delta_{el}, \delta_{er}, \delta_a, \delta_r]^T$. The velocity, angle of attack and side slip angle are given by $V = \sqrt{u^2 + v^2 + w^2}$, $\alpha = \arctan \frac{w}{u}$ and $\beta = \arcsin \frac{v}{V}$ respectively.

4. CONTROLLER DESIGN

The aircraft's dynamics are inherently unstable and must be augmented with a stabilizing controller, viz, a Stability Augmentation System (SAS). The primary objective of the controller is to stabilize the system about its operating point, and also to have a degree of robustness to enable turns, and possess acceptable handling qualities.

The closed loop eigenvalues are based on the guidelines provided in (Rynaski, 1982) and (Pamadi, 1998). The multivariable nature of the system gives us some flexibility in choosing the eigenvectors as well (Moore, 1976). The decoupling between longitudinal and lateral dynamics that is inherent in the level flight condition is preserved in the design.

The Dutch roll motion is a "flat" yawing/sideslipping motion in which rolling is suppressed (Etkin and Reid, 1996). Hence we ensure that ϕ is not excited in the Dutch roll mode. In both the spiral and the roll subsidence modes there is negligible sideslip and so in these modes the eigenvector is tailored not to affect v. The heading mode inherently does not influence p and r. Among the states it influences, namely ϕ, ψ and v, the eigenvector is designed so that in this mode only ψ is affected.

The thrust is left to the pilot. It was found that the closed loop system was very susceptible to becoming unstable if thrust was part of the feedback.

The SAS was implemented as a linear state feedback control law as

$$\begin{aligned}
T &= T_c \\
\delta_{el} &= \delta_{elc} - 1.3945 + 1.2722q + 0.3398\theta \\
\delta_{er} &= \delta_{erc} - 1.3945 + 1.2722q + 0.3398\theta \\
\delta_a &= \delta_{ac} + 6.4660p + 0.4058\phi - 3.1594r - 0.03101v \\
\delta_r &= \delta_{rc} - 4.93p + 2.2151\phi - 25.4988r - 0.1877v
\end{aligned}$$

where T_c, δ_{elc}, δ_{erc}, δ_{ac}, δ_{rc} denote the pilot's commanded inputs.

5. LEVEL FLIGHT BIFURCATION ANALYSIS

In level flight, the body fixed frame has no rotational component with respect to the earth fixed frame and thus the roll ϕ, flight path angle Γ and heading Ψ are all zero. In this work, a positive ϕ and positive Ψ are along the positive X and positive Z_E directions respectively, while a positive Γ is in the negative Y_E direction. The subscript E is used to denote the earth fixed axis.

In level flight we are interested in regulating the velocity and orientation of the aircraft, i.e.,

$$\mathbf{r}(x,\, \mu) = \{V,\, \phi,\, \Gamma,\, \Psi\}$$

The 4 regulated variables, together with the 9 state equations result in 13 equations. There are 14 variables, namely: ϕ, θ, ψ, p, q, r, V, α, β, T, δ_{el}, δ_{er}, δ_a, δ_r. The velocity V is the parameter. In level flight, the variables ϕ, ψ, p, q, r, β, δ_a, δ_r are trivially zero. Also, $\delta_{el} = \delta_{er}$. The bifurcation diagram is shown in Figure 1. Although the fold bifurcations can be observed in the bifurcation diagrams of the other variables as well, only δ_{el} is chosen due to space constraints.

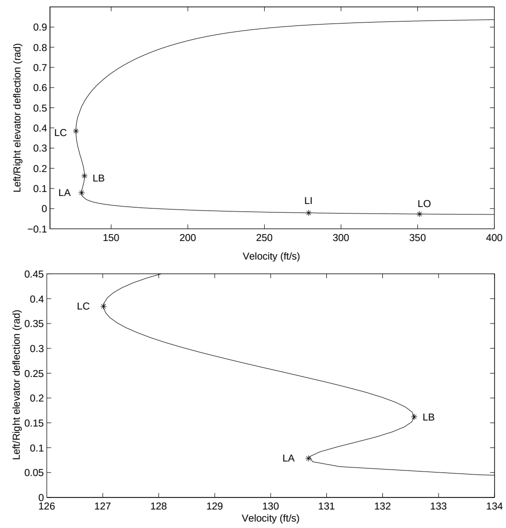

Fig. 1. Level flight bifurcation diagrams.

The closed loop bifurcation analysis is basically the same as that for the open loop case. The bifurcation curves of the pilot's commanded inputs can be determined from equation (4) once the open loop analysis is complete. However the analysis of stability and system characteristics at the bifurcation points are different for the open and closed loop cases.

It was found that at all three bifurcation points the open and closed loop systems are unstable. At the bifurcation points for both the open and closed loop case, the linear system has transmission zeros at the origin, is uncontrollable and has dependent inputs.

In level flight, at the bifurcation points the aircraft usually stalls. This was true of all the bifurcation points in both the open and closed loop configurations, except at bifurcation points LA and LB in the open loop scenario. We omit figures that illustrate simple stall because of space limitations. As time progresses, the angle of attack increases causing the aircraft to stall. The aircraft, in a nose up configuration is headed downwards and loses altitude.

At bifurcation points LA and LB in the open loop configuration, the aircraft experiences the tumbling stall phenomenon. The dynamics are qualitatively similar at the two point and thus only the simulations for LA are shown in Figure 2. Although the Euler angles are based on a 3-2-1 convention and could cause ambiguity when the aircraft is pointed vertically up or down, this is not the case here since both the roll angle ϕ and yaw angle ψ are zero.

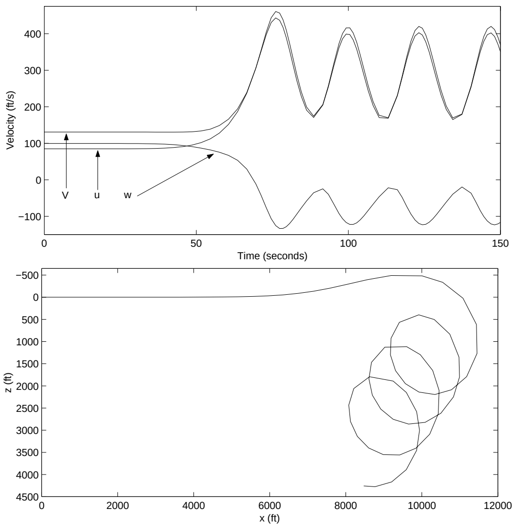

Fig. 2. Dynamics of the open loop system at bifurcation point LA.

6. COORDINATED TURN BIFURCATION ANALYSIS

A coordinated turn is one that satisfies the following two conditions (Etkin and Reid, 1996):

C1. The angular velocity of the aircraft is constant and vertical.

C2. The resultant of the gravity and centrifugal force at the center of gravity lies in the plane of symmetry (the x-z plane) of the aircraft.

These conditions lead to the constraint

$$\mathcal{L} := r\,u - p\,w - g\cos\theta\,\sin\phi = 0$$

We obtain an equilibrium condition corresponding to a coordinated turn at level flight by starting at equilibrium point LO and employing the continuation approach. The 6 dynamic equations together with the coordinated turn and constant altitude condition result in 8 equilibrium equations. There are 12 variables, namely, ϕ, θ, ψ ,V, α, β, T, δ_{el}, δ_{er}, δ_a, δ_r, ω. We hold V, ψ, δ_{el}, δ_{er} fixed and vary ω, until we reached a radius ($R = V/\omega = -V\,\sin\theta/p$) of 8012.2 ft. The choice of the radius was arbitrary.

The regulated variables are

$$\mathbf{r}(x,\,\mu) = \{V,\,\mathcal{L},\,\Gamma,\,R\}$$

Again, the 6 dynamic equations together with the coordinated turn and constant altitude conditions result in 8 equilibrium equations. There are 10 variables namely: ϕ, θ, V, α, β, T, δ_{el}, δ_{er}, δ_a and δ_r. We set $\delta_{el} = \delta_{er}$ and treat V as the parameter.

As in the level flight scenario, there were three bifurcation points. At all three points the open and closed loop systems were unstable. At all the bifurcation points, the system was uncontrollable, unobservable, had dependent controls, dependent regulated variables and a transmission zero at the origin, for both the open and closed loop. The bifurcation diagram is shown in Figure 3.

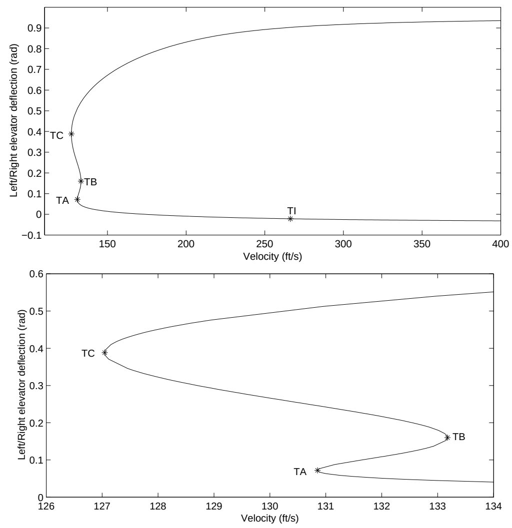

Fig. 3. Bifurcation diagrams for a coordinated turn of radius 8012.2 ft.

As in the level flight case the aircraft stall at all bifurcation points in the coordinated turn case except at bifurcation point TB in the open loop scenario. At bifurcation point TB, initially the aircraft appears to enter a spin. However it gradually deviates from the spin and comes down in a fast roll. The path taken by the vehicle is shown in Figure 4. It should be noted that the aircraft is constantly rolling about its axis. For the other bifurcation points the aircraft appears to stall. There is no indication of it entering into a spin motion.

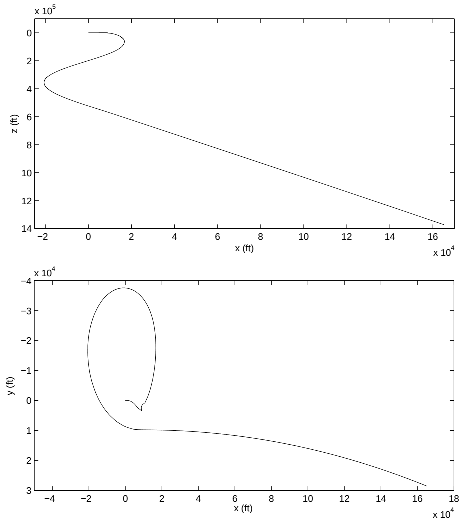

Fig. 4. Dynamics of the open loop system at bifurcation point TB.

7. CONCLUSIONS

The bifurcations occurring in an F-16 in straight and level flight, and in a coordinated turn were identified and characterized. Three bifurcation points were identified in each case. The bifurcations are associated with the regulation of the aircraft's speed and orientation. The speeds at which the bifurcations occur do not differ substantially between the two cases. The longitudinal variables also do not show much difference. In straight and level flight, at the bifurcation points in both the compensated and uncompensated cases, there is loss of controllability, the presence of dependent inputs and a transmission zero at the origin. At the bifurcation points encountered in a coordinated turn or radius 8018.2 ft, the aircraft also loses observability and has redundant outputs.

In most cases the bifurcations encountered were stalls. However, stall was followed by tumbling in one instance of a bifurcation in straight and level flight. During a coordinated turn the vehicle was observed to stall, enter a spin and then exit to roll divergence before the spin was fully developed.

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